

A FREE-BOUNDARY PROBLEM FOR THE EVOLUTION p -LAPLACIAN EQUATION WITH A COMBUSTION BOUNDARY CONDITION

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ABSTRACT. We study the existence, uniqueness and regularity of solutions of the equation $f_t = \Delta_p f = \operatorname{div}(|Df|^{p-2} Df)$ under over-determined boundary conditions $f = 0$ and $|Df| = 1$. We show that if the initial data is concave and Lipschitz with a bounded and convex support, then the problem admits a unique solution which exists until it vanishes identically. Furthermore, the free-boundary of the support of f is smooth for all positive time.

1. INTRODUCTION

Fix a number $p > 2$. Given a non-negative function f_0 on \mathbb{R}^n with positive set Ω_0 , we want to find a non-negative function $f(x, t)$ on $\mathbb{R}^n \times (0, T)$ with positive set Ω which solves the following problem:

$$(P) \quad \begin{cases} f_t = \Delta_p f & \text{in } \Omega = \{f > 0\} \\ f = 0 \text{ and } |Df| = 1 & \text{on } \partial\Omega \cap \{0 < t < T\} \\ \lim_{t \rightarrow 0} f(\mathbf{x}, t) = f_0(\mathbf{x}) & \forall \mathbf{x} \in \mathbb{R}^n. \end{cases}$$

The operator

$$\Delta_p f = \operatorname{div}(|Df|^{p-2} Df)$$

is known as the p -Laplacian. In non-divergent form, it can be written as

$$(1.1) \quad \Delta_p = |Df|^{p-2} \Delta f + (p-2)|Df|^{p-4} f_{ij} f_i f_j$$

Note that the Einstein summation notation was used in the last term. It can also be written as

$$(1.2) \quad \Delta_p = |Df|^{p-2} (\Delta f + (p-2) f_{\nu\nu})$$

where $f_{\nu\nu}$ denotes the second derivative of f in the direction of $\nu = Df/|Df|$.

In the case $p > 2$, this operator is nonlinear and degenerate at vanishing points of Df . When $p = 2$, it is just the regular Laplacian.

2000 *Mathematics Subject Classification.* Primary: 35R35; Secondary: 35K55, 35K65.

Key words and phrases. p -Laplacian, free-boundary problem, degenerate equation, combustion, regularization, convex domain.

Due to the over-determined boundary conditions $f = 0$ and $|Df| = 1$, the time-section of Ω

$$\Omega_t = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}, t) > 0\}$$

will in general change with time. In other words, the boundary $\partial\Omega_t$ moves. It is often known as the moving-boundary or free-boundary.

Our work is motivated by the work of Caffarelli and Vázquez [2] in which authors studied this problem in the case $p = 2$. Their result stated essentially that if $\partial\Omega_0 \in C^2$, $f_0 \in C^2(\Omega_0)$ and $\Delta f_0 \leq 0$, then there exists a solution to the problem. Moreover, if Ω_0 is compact, solutions vanish in finite time. Still in this case, long-time existence, uniqueness and regularity of the free-boundary have been studied by Daskalopoulos and Ki-Ahm Lee [5] or Petrosyan [11, 12] when the initial value is concave or star-shaped with bounded support. Other kinds of solution have also been studied (see also [9]). In the case $p > 2$, an elliptic version of the problem has been studied before by Danielli, Petrosyan and Shahgholian [3] or Henrot and Shahgholian [7, 8]. As far as the parabolic problem when $p > 2$ is concerned, the only result we are aware of is by Akopyan and Shahgholian [1] where authors showed the uniqueness under the hypotheses that the time-section Ω_t is convex and non-decreasing in time. The questions of existence or regularity of the free-boundary were not addressed in that paper.

The main result of our work is stated below.

Theorem 1.1. *Assume that Ω_0 is a bounded and convex domain. The function f_0 is positive and concave in Ω_0 . Furthermore, on the boundary $\partial\Omega_0$, f_0 satisfies*

$$\begin{aligned} f_0(\mathbf{x}) &= 0 \quad \text{for all } \mathbf{x} \\ |Df_0(\mathbf{x})| &= 1 \quad \text{for a.e. } \mathbf{x}. \end{aligned}$$

Then the problem (P) has a unique solution up to a finite time T where it vanishes identically in the sense that

$$\lim_{t \rightarrow T} f(\mathbf{x}, t) = 0 \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the free-boundary $\partial\Omega_t$ is smooth for all $t \in (0, T)$.

It is well-known that solutions of the evolution p -Laplacian are only $C^{1,\alpha}$ at points of vanishing gradient (see for example [6]). Hence, solutions to the problem (P) must be defined in some weak sense. We will state precisely the meaning of our solution in section 2.

Our approach to the problem is totally different from [2]. To deal with the degeneracy, we will approximate the p -Laplacian with the following regularized operator

$$(P(\epsilon)) \quad \Delta_p^\epsilon f = \operatorname{div}((|Df|^2 + \epsilon)^{q-1} Df).$$

Here and throughout this work, we define $q = p/2$. We will establish some properties for solutions of these regularized problems and then let ϵ go to 0 to obtain a solution to the degenerate problem.

In order to solve this regularized free-boundary problem, we employ a change of coordinates that transforms it into a quasilinear equation with Neumann boundary condition on

a fixed-domain problem. Applying results from standard theory of quasi-linear parabolic equations with oblique boundary condition, we show that this new problem admits a solution for some positive time. Revert back to the original coordinates, we obtain a short-time existence result for the regularized problem. This argument is carried out in section 3.

In section 4, we prove a simple estimate for the gradient $|Df|$ of solutions of the problem $(P(\epsilon))$. In section 5, we prove a crucial result that the time-section Ω_t remains convex and the function $f(., t)$ remains concave on Ω_t for all time t . Convexity of Ω_t guarantees that the free-boundary $\partial\Omega_t$ does not touch itself and also enables us to prove the non-degeneracy of $|Df|$ near the free-boundary.

In section 6, we obtain an estimate for higher derivatives of f in a neighborhood the free-boundary $\partial\Omega_t$, uniformly in time t and especially, in ϵ , using the non-degeneracy of $|Df|$. This fact and the convexity guarantee that singular cannot develop on the free-boundary. The uniqueness for this regularized problem is obtained in section 7. In section 8, we then obtain a long-time existence result for solution of the regularized problem. Passing ϵ to 0, we then obtain a solution to the degenerate problem in section 9. The uniqueness for the degenerate problem is then shown in section 10. In the last section, we show that solution to our degenerate problem vanishes in finite time.

Acknowledgement. I express my gratitude to my thesis advisor, P. Daskalopoulos, for suggesting this problem, and for her invaluable advices and support during the completion of this work.

2. DEFINITION OF SOLUTION

In this section, we will define precisely what we mean by solution of the problem (P). We start by introducing some notations. For any $0 < t_1 < t_2 < T$, define

$$\Omega_{(t_1, t_2)} = \Omega \cap \{t_1 < t < t_2\}.$$

First, we require that the free-boundary $\partial\Omega_t$ is in C^1 and the function f is in

$$C(0, T; C^1(\overline{\Omega_t})).$$

The equation

$$f_t = \Delta_p f \quad \text{in } \Omega$$

is then defined in the sense that for any test function θ in $C_0^\infty(\Omega)$ and for any $0 < t_1 < t_2 < T$,

$$\int_{\Omega_{(t_1, t_2)}} f \theta_t \, d\mathbf{x} dt - \int_{\Omega_{(t_1, t_2)}} f \theta \, d\mathbf{x} \Big|_{\Omega_{t_1}}^{\Omega_{t_2}} = \int_{\Omega_{(t_1, t_2)}} |Df|^{p-2} Df \cdot D\theta \, d\mathbf{x} dt.$$

The Cauchy-Dirichlet conditions $f = 0$ on $\partial\Omega_t$ and $f(., 0) = f_0$ are understood in the pointwise sense

$$\begin{aligned} f(\mathbf{x}, t) &\rightarrow 0 \quad \text{as } \mathbf{x} \rightarrow \mathbf{x}_0 \in \partial\Omega_t, \\ f(\mathbf{x}, t) &\rightarrow f_0(\mathbf{x}) \quad \text{as } t \rightarrow 0. \end{aligned}$$

Finally, the Neumann's boundary condition $|Du| = 1$ is defined in the following classical sense

$$f_\nu(\mathbf{x}_0, t) = \lim_{h \rightarrow 0^+} \frac{f(\mathbf{x}_0 + h\nu)}{h} = 1.$$

where \mathbf{x}_0 is a point on the free-boundary $\partial\Omega_t$ and ν is the spatial inward unit normal vector at \mathbf{x}_0 with regards to $\partial\Omega_t$.

3. SHORT-TIME EXISTENCE FOR REGULARIZED PROBLEM

In this section, we will prove that the regularized free-boundary problem admits a solution for some positive time. We do it by a change of coordinates technique that transforms the problem into a fixed-domain problem. This technique has been used by other authors for different problems before (see for example [5], [4]). Note that concavity is not needed in this result.

Lemma 3.1. *Assume that Ω_0 is C^∞ . The function f_0 is in $C^\infty(\overline{\Omega_0})$ and positive in Ω_0 . Furthermore, on the boundary $\partial\Omega_0$, f_0 satisfies*

$$f_0 = 0 \quad \text{and} \quad |Df_0| = 1.$$

Then there exists a smooth solution to the regularized problem (P(ϵ)) for some $T > 0$.

Proof. The argument in this proof works for any dimension, but due to the complexity of some computation involved, we will present the proof for the case $n = 2$ only.

A word on notation used in this proof : we use bold-face letters $\mathbf{x}, \mathbf{y}, \dots$ to denote points in Euclidean spaces while normal letters x, y, z, \dots for real numbers, scalars or components of points in Euclidean spaces.

Denote by \mathcal{S} the smooth surface $z = f_0(x, y)$, $(x, y) \in \overline{\Omega_0}$. Let $T = (T_1, T_2, T_3)$ be a smooth vector field on $\overline{\Omega_0}$ such that $T(x, y)$ is not a tangential vector to the surface \mathcal{S} at the point $f_0(x, y)$. Since $|Df_0| = 1$ on the boundary $\partial\Omega_0$, we can also choose T to be parallel to the plane $z = 0$ in a small neighborhood of $\partial\Omega_0$.

It is known that for some positive, small enough η , we can define a change of spatial coordinates

$$\Phi : \Omega_0 \times [-\eta, \eta] \rightarrow \mathbb{R}^3$$

by the formula

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi \begin{pmatrix} u \\ v \\ w \end{pmatrix} = f_0 \begin{pmatrix} u \\ v \end{pmatrix} + w T \begin{pmatrix} u \\ v \end{pmatrix}.$$

The map Φ defines x, y and z as smooth functions of u, v and w with smooth inverses.

The graph of $(x, y, f(x, y, t))$, $(x, y) \in \Omega_t$ is then transformed to $(u, v, g(u, v, t))$, $(u, v) \in \Omega_0$ via this coordinates change for some uniquely-defined g if the surface $z = f(x, y, t)$ is sufficiently close to \mathcal{S} ($(x, y, f(x, y, t)) \in \Phi(\Omega_0 \times [-\eta, \eta])$ for all $(x, y) \in \Omega_t$). When f evolves as a function of (x, y) , g evolves as a function of (u, v) . Importantly, the domain of g is fixed as Ω_0 due to our requirement that T is parallel to the plane $z = 0$ on $\partial\Omega_0$.

We will compute the evolution equation and the boundary condition of g . Denote by $x_u, x_v, x_w, y_u, y_v, y_w, z_u, z_v$ and z_w the partial derivatives of the functions $x(u, v, w)$, $y(u, v, w)$ and $z(u, v, w)$. Similarly we denote partial second derivatives of x, y and z by x_{uu}, x_{uv}, \dots

We begin with first derivatives. Since x, y and z are functions of u, v and w , while $w = g(u, v, t)$ is a function of u, v and t , we have

$$\begin{aligned} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} &= \begin{pmatrix} x_u + x_w \frac{\partial w}{\partial u} & y_u + y_w \frac{\partial w}{\partial u} \\ x_v + x_w \frac{\partial w}{\partial v} & y_v + y_w \frac{\partial w}{\partial v} \end{pmatrix} \\ &= \begin{pmatrix} x_u + x_w g_u & y_u + y_w g_u \\ x_v + x_w g_v & y_v + y_w g_v \end{pmatrix}. \end{aligned}$$

We can compute the partial derivatives of $u(x, y, t)$ and $v(x, y, t)$ by

$$(3.1) \quad \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}^{-1} = \frac{1}{D} \begin{pmatrix} \frac{\partial y}{\partial v} & -\frac{\partial x}{\partial v} \\ -\frac{\partial y}{\partial u} & \frac{\partial x}{\partial u} \end{pmatrix}$$

$$(3.2) \quad = \frac{1}{D} \begin{pmatrix} y_v + y_w g_v & -x_v - x_w g_v \\ -y_u - y_w g_u & x_u + x_w g_u \end{pmatrix}$$

where

$$D = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = (x_u y_v - x_v y_u) + (x_w y_v - y_w x_v) g_u + (y_w x_u - x_w y_u) g_v.$$

We then have

$$(3.3) \quad \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial z}{\partial u} \\ \frac{\partial z}{\partial v} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} \frac{\partial y}{\partial v} & -\frac{\partial x}{\partial v} \\ -\frac{\partial y}{\partial u} & \frac{\partial x}{\partial u} \end{pmatrix} \begin{pmatrix} z_u + z_w g_u \\ z_v + z_w g_v \end{pmatrix}.$$

Next we compute the second-order derivatives. First, we have partial second order derivatives of x with regards to u and v .

$$\begin{aligned} \frac{\partial^2 x}{\partial^2 u} &= x_{uu} + 2x_{uw} \frac{\partial w}{\partial u} + x_{ww} \left(\frac{\partial w}{\partial u} \right)^2 + x_w \frac{\partial^2 w}{\partial^2 u} \\ &= x_{uu} + 2x_{wu} g_u + x_w g_{uu} \\ \frac{\partial^2 x}{\partial^2 v} &= x_{vv} + 2x_{vw} g_v + x_w g_{vv} \\ \frac{\partial^2 x}{\partial u \partial v} &= x_{uv} + x_{wu} g_v + x_{vw} g_u + x_w g_{uv} \end{aligned}$$

and similar formulae for y and z .

Differentiate (3.3) we have

$$(3.4) \quad f_{xx} = \frac{\partial^2 z}{\partial^2 x} = \frac{\partial z}{\partial u} \frac{\partial^2 u}{\partial^2 x} + \frac{\partial z}{\partial v} \frac{\partial^2 v}{\partial^2 x} + \frac{\partial^2 z}{\partial^2 u} \left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 z}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial^2 v} \left(\frac{\partial v}{\partial x} \right)^2.$$

We need to compute second order derivatives of u and v with regards to x and y . The formula (3.4) is true if we substitute any function of u and v in place of z . Because second order derivatives of x and y with regards to x are zero

$$\begin{aligned} 0 &= \frac{\partial x}{\partial u} \frac{\partial^2 u}{\partial^2 x} + \frac{\partial x}{\partial v} \frac{\partial^2 v}{\partial^2 x} + \frac{\partial^2 x}{\partial^2 u} \left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 x}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 x}{\partial^2 v} \left(\frac{\partial v}{\partial x} \right)^2 \\ 0 &= \frac{\partial y}{\partial u} \frac{\partial^2 u}{\partial^2 x} + \frac{\partial y}{\partial v} \frac{\partial^2 v}{\partial^2 x} + \frac{\partial^2 y}{\partial^2 u} \left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 y}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial^2 v} \left(\frac{\partial v}{\partial x} \right)^2. \end{aligned}$$

In other words

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 u}{\partial^2 x} \\ \frac{\partial^2 v}{\partial^2 x} \end{pmatrix} + \begin{pmatrix} \frac{\partial^2 x}{\partial^2 u} \left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 x}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 x}{\partial^2 v} \left(\frac{\partial v}{\partial x} \right)^2 \\ \frac{\partial^2 y}{\partial^2 u} \left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 y}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial^2 v} \left(\frac{\partial v}{\partial x} \right)^2 \end{pmatrix} = 0$$

or

$$\begin{aligned} \begin{pmatrix} \frac{\partial^2 u}{\partial^2 x} \\ \frac{\partial^2 v}{\partial^2 x} \end{pmatrix} &= - \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial^2 x}{\partial^2 u} \left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 x}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 x}{\partial^2 v} \left(\frac{\partial v}{\partial x} \right)^2 \\ \frac{\partial^2 y}{\partial^2 u} \left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 y}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial^2 v} \left(\frac{\partial v}{\partial x} \right)^2 \end{pmatrix} \\ &= - \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 x}{\partial^2 u} \left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 x}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 x}{\partial^2 v} \left(\frac{\partial v}{\partial x} \right)^2 \\ \frac{\partial^2 y}{\partial^2 u} \left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 y}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial^2 v} \left(\frac{\partial v}{\partial x} \right)^2 \end{pmatrix}. \end{aligned}$$

We then have

$$\begin{aligned} \frac{\partial z}{\partial u} \frac{\partial^2 u}{\partial^2 x} + \frac{\partial z}{\partial v} \frac{\partial^2 v}{\partial^2 x} &= \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 u}{\partial^2 x} \\ \frac{\partial^2 v}{\partial^2 x} \end{pmatrix} \\ &= - \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 x}{\partial^2 u} \left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 x}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 x}{\partial^2 v} \left(\frac{\partial v}{\partial x} \right)^2 \\ \frac{\partial^2 y}{\partial^2 u} \left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 y}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial^2 v} \left(\frac{\partial v}{\partial x} \right)^2 \end{pmatrix} \\ &= - \begin{pmatrix} f_x & f_y \end{pmatrix} \begin{pmatrix} \frac{\partial^2 x}{\partial^2 u} \left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 x}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 x}{\partial^2 v} \left(\frac{\partial v}{\partial x} \right)^2 \\ \frac{\partial^2 y}{\partial^2 u} \left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 y}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 y}{\partial^2 v} \left(\frac{\partial v}{\partial x} \right)^2 \end{pmatrix}. \end{aligned}$$

Substitute into (3.4)

$$\begin{aligned} f_{xx} &= \left(\frac{\partial^2 z}{\partial^2 u} - f_x \frac{\partial^2 x}{\partial^2 u} - f_y \frac{\partial^2 y}{\partial^2 u} \right) \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial^2 z}{\partial^2 v} - f_x \frac{\partial^2 x}{\partial^2 v} - f_y \frac{\partial^2 y}{\partial^2 v} \right) \left(\frac{\partial v}{\partial x} \right)^2 \\ &\quad + 2 \left(\frac{\partial^2 z}{\partial u \partial v} - f_x \frac{\partial^2 x}{\partial u \partial v} - f_y \frac{\partial^2 y}{\partial u \partial v} \right) \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}. \end{aligned}$$

Let

$$\begin{aligned}
E &= z_w - f_x x_w - f_y y_w \\
A &= \frac{\partial^2 z}{\partial^2 u} - f_x \frac{\partial^2 x}{\partial^2 u} - f_y \frac{\partial^2 y}{\partial^2 u} \\
&= E g_{uu} + 2(z_{wu} - f_x x_{wu} - f_y y_{wu}) g_u + (z_{uu} - f_x x_{uu} f_y y_{uu}) \\
B &= \frac{\partial^2 z}{\partial^2 v} - f_x \frac{\partial^2 x}{\partial^2 v} - f_y \frac{\partial^2 y}{\partial^2 v} \\
&= E g_{vv} + 2(z_{wv} - f_x x_{wv} - f_y y_{wv}) g_v + (z_{vv} - f_x x_{vv} f_y y_{vv}) \\
C &= \frac{\partial^2 z}{\partial u \partial v} - f_x \frac{\partial^2 x}{\partial u \partial v} - f_y \frac{\partial^2 y}{\partial u \partial v} \\
&= E g_{uv} + (z_{wv} - f_x x_{wv} - f_y y_{wv}) g_u + (z_{wu} - f_x x_{wu} - f_y y_{wu}) g_v \\
&\quad + (z_{uv} - f_x x_{uv} - f_y y_{uv}),
\end{aligned}$$

then

$$\begin{aligned}
f_{xx} &= A \left(\frac{\partial u}{\partial x} \right)^2 + B \left(\frac{\partial v}{\partial x} \right)^2 + 2C \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \\
&= E \left(\left(\frac{\partial u}{\partial x} \right)^2 g_{uu} + \left(\frac{\partial v}{\partial x} \right)^2 g_{vv} + 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} g_{uv} \right) + \frac{F}{D^2}
\end{aligned}$$

where F is a smooth function of u, v, g, g_u, g_v . We have similar formulae for f_{xy} and f_{yy}

$$\begin{aligned}
f_{yy} &= E \left(\left(\frac{\partial u}{\partial y} \right)^2 g_{uu} + \left(\frac{\partial v}{\partial y} \right)^2 g_{vv} + 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} g_{uv} \right) + \frac{F}{D^2} \\
f_{xy} &= E \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} g_{uu} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} g_{vv} + \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \right) g_{uv} \right) + \frac{F}{D^2}
\end{aligned}$$

where F denotes different smooth functions of (u, v, g, g_u, g_v) .

To compute f_t , we differentiate $z = f(x, y, t)$

$$\begin{aligned}
z_w \frac{\partial w}{\partial t} &= f_t + (f_x x_w + f_y y_w) w_t \\
f_t &= (z_w - f_x x_w - f_y y_w) g_t = E g_t.
\end{aligned}$$

Substituting into the equation for f

$$f_t = (|Df|^2 + \epsilon)^{q-1} \Delta f + (p-2)(|Df|^2 + \epsilon)^{q-2} (f_{xx} f_x^2 + f_{yy} f_y^2 + 2f_{xy} f_x f_y)$$

and simplifying E from both sides we then obtain an evolution equation for g in the form

$$g_t = A^{ij}(u, v, g, Dg) g_{ij} + B(u, v, g, Dg).$$

On the other hand, the boundary condition $|Df| = 1$ becomes

$$C(u, v, g, Dg) = 0$$

for some function C .

We claim the following is true when $g \equiv 0$ (i.e at $t = 0$) :

- A^{ij}, B and C are smooth functions of u, v, g and Dg .
- (A^{ij}) is positive definite.
- C is oblique.

Because the surface \mathcal{S} and the vector field T are both smooth, it is clear that A^{ij}, B and C are smooth functions of u, v, g and Dg whenever

$$D = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \neq 0$$

$$E = z_w - f_x x_w - f_y y_w \neq 0.$$

The condition that $E \neq 0$ follows from our choice that T is transverse to \mathcal{S} . The condition $D \neq 0$ is a consequence of the fact that the function Φ is invertible in a neighborhood of \mathcal{S} .

Next, to show that (A^{ij}) is positive definite, we write

$$A^{ij} = (|Df|^2 + \epsilon)^{q-1} A_1^{ij} + (p-2)(|Df|^2 + \epsilon)^{q-2} A_2^{ij}$$

where A_1^{ij} is the coefficient of g_{ij} ($i, j \in \{u, v\}$) obtained from the transformation of Δf and A_2^{ij} from $f_x^2 f_{xx} + f_y^2 f_{yy} + 2f_x f_y f_{xy}$. We can compute explicitly

$$A_1 = \begin{pmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 & \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} & \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \left(f_x \frac{\partial u}{\partial x} + f_y \frac{\partial u}{\partial y}\right)^2 & \left(f_x \frac{\partial u}{\partial x} + f_y \frac{\partial u}{\partial y}\right) \left(f_x \frac{\partial v}{\partial x} + f_y \frac{\partial v}{\partial y}\right) \\ \left(f_x \frac{\partial u}{\partial x} + f_y \frac{\partial u}{\partial y}\right) \left(f_x \frac{\partial v}{\partial x} + f_y \frac{\partial v}{\partial y}\right) & \left(f_x \frac{\partial v}{\partial x} + f_y \frac{\partial v}{\partial y}\right)^2 \end{pmatrix}.$$

It is obvious that A_1 and A_2 are non-negative definite. Furthermore, if $D \neq 0$, A_1 is actually positive definite ($\det(A_1) = D^2$). It then follows readily that (A^{ij}) is positive definite.

For the proof that C is oblique, we refer to the Appendix of [5].

From the continuity, there must exist a positive number δ such that those three claims are true for all g that satisfies $|g|_{C^1(\Omega_0)} < \delta$. It is then a consequence of standard theory of quasilinear parabolic equation with oblique boundary condition (see for examples [10], Chapter 14) that there exists a solution g up to a positive time T to the problem.

$$\begin{cases} g_t = A^{ij} g_{ij} + B & \text{in } \Omega_0 \times (0, T) \\ C(u, v, g, Dg) = 0 & \text{on } \partial\Omega_0 \times (0, T) \\ g(\cdot, 0) = 0. \end{cases}$$

This solution is actually smooth up to the boundary for all $t \in [0, T)$ since Ω_0 is smooth and $C(u, v, g, Dg)$ is a smooth function of (u, v, g, Dg) . Choose a number T' in $(0, T]$ such that $|g| < \eta$ on $\Omega_0 \times (0, T')$. Reverting back to the original coordinates system we then

obtain a solution to the regularized problem $(P(\epsilon))$ up to T' . It is clear that the domain Ω_t is smooth and the solution f is smooth up to the free-boundary for all time $0 < t < T'$. \square

4. GRADIENT ESTIMATE

Lemma 4.1. *Assume the same hypotheses as in the Lemma 3.1. Furthermore, f_0 satisfies $|Df_0| \leq 1$ on Ω_0 . If f is a solution of the problem $(P(\epsilon))$, then*

$$|Df(\mathbf{x}, t)| < 1$$

for all $(\mathbf{x}, t) \in \Omega$.

Proof. We will show an equivalent fact that

$$f_\lambda(\mathbf{x}, t) < 1$$

for any unit vector λ .

Let

$$a^{ij}(Df) = (|Df|^2 + \epsilon)^{q-1} \delta_{ij} + 2(q-1)(|Df|^2 + \epsilon)^{q-2} f_i f_j$$

where δ_{ij} is the Kronecker delta function. Recall that we define $q = p/2$ throughout this work. Then the evolution equation of f can be written in non-divergent form as

$$\begin{aligned} f_t &= (|Df|^2 + \epsilon)^{q-1} \Delta f + 2(q-1)(|Df|^2 + \epsilon)^{q-2} f_{ij} f_i f_j \\ &= a^{ij} f_{ij}. \end{aligned}$$

We compute the evolution equation for f_λ

$$f_{\lambda t} = a^{ij} f_{\lambda ij} + (f_{ij} D a^{ij}) D f_\lambda.$$

Since this equation satisfies the Strong Maximum Principle, f_λ must attain its maximum value on the parabolic boundary of Ω . Because $f_\lambda \leq 1$ on the parabolic boundary of Ω , it then follows that

$$f_\lambda < 1$$

in Ω for all unit vector λ . \square

Lemma 4.2. *Assume the same as in the last lemma, then at any point \mathbf{x}_0 on the free-boundary $\partial\Omega_t$*

$$f_{\nu\nu}(\mathbf{x}_0, t) < 0$$

where ν is the inward normal vector at \mathbf{x}_0 with regards to $\partial\Omega_t$.

Proof. Apply Hopf's Lemma to the evolution equation for f_ν from the last lemma, observing that f_ν attains the maximum value of 1 at (\mathbf{x}_0, t) . \square

5. CONVEXITY

In this section we will show that the time-section Ω_t remains convex and the function $f(., t)$ remains concave on Ω_t . Normally, for this kind of question, the main difficulty lies in showing that Ω_t remains convex. The arguments for the case $p = 2$ as in [11] or [5] do not translate directly to the case $p > 2$. On the other hand, our argument here can be simplified to give a new and simple proof for the case $p = 2$. The argument relies heavily on the Neumann boundary condition $|Df| = 1$.

Lemma 5.1. *Assume the same hypotheses as in the Lemma 3.1. Furthermore, assume that Ω_0 is strictly convex and f_0 is strictly concave on Ω_0 . If f is a solution to the problem (P(ϵ)) up to some positive time T , then Ω_t is strictly convex and $f(., t)$ is strictly concave for all $t \in [0, T)$.*

Proof. We will show that

$$f_{\lambda\lambda}(\mathbf{x}, t) < 0$$

for any point $(\mathbf{x}, t) \in \overline{\Omega}$, any unit vector λ and any $t \in [0, T']$ where T' is any number strictly less than T . Clearly this implies that Ω_t is strictly convex and f is strictly concave for all $t \in [0, T)$.

First, we compute the evolution equation of $f_{\lambda\lambda}$,

$$\begin{aligned} f_t &= a^{ij} f_{ij} \\ f_{\lambda t} &= a^{ij} f_{\lambda ij} + (f_{ij} D a^{ij}) \cdot D f_\lambda \\ f_{\lambda\lambda t} &= a^{ij} f_{\lambda\lambda ij} + 2(f_{\lambda ij} D a^{ij}) \cdot D f_\lambda + (f_{ij} D a^{ij}) \cdot D f_{\lambda\lambda} + (f_{ij} (D a^{ij})_\lambda) \cdot D f_\lambda \\ &= a^{ij} f_{\lambda\lambda ij} + (f_{ij} D a^{ij}) \cdot D f_{\lambda\lambda} + (2f_{\lambda ij} D a^{ij} + f_{ij} D (a^{ij})_\lambda) \cdot D f_\lambda \end{aligned}$$

Since f is smooth for all $t \in (0, T)$, there exists a finite number $C(T')$ such that for any unit vector λ and any point $(\mathbf{x}, t) \in \overline{\Omega} \cap \{0 < t \leq T'\}$

$$|2f_{\lambda ij} D a^{ij} + f_{ij} D (a^{ij})_\lambda| < C.$$

Choose a smooth function v_0 on \mathbb{R}^n such that

$$\begin{cases} 0 < v_0 < -(f_0)_{\lambda\lambda} & \text{in } \Omega_0 \\ v_0 = 0 & \text{on } \partial\Omega_0. \end{cases}$$

Such v_0 exists because f_0 is strictly concave on Ω_0 . Let v be the solution of the Cauchy-Dirichlet problem

$$\begin{cases} v_t = a^{ij} v_{ij} + (f_{ij} D a^{ij}) \cdot D v - C v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \cap \{0 < t < T\} \\ v(., 0) = v_0 & \text{in } \Omega_0. \end{cases}$$

Applying Strong Maximum Principle and Hopf's Lemma to v we easily deduce that

$$\begin{cases} v > 0 & \text{in } \Omega \\ |Dv| > 0 & \text{on } \partial\Omega \cap \{0 < t < T\}. \end{cases}$$

We are going to show that

$$(5.1) \quad v + f_{\lambda\lambda} < 0$$

for all $t \in [0, T']$ and all unit vector λ . Assuming that it is not the case, i.e there exists some point (\mathbf{x}', t') and some unit vector λ' such that

$$(v + f_{\lambda'\lambda'}) (\mathbf{x}', t') = 0$$

and

$$v + f_{\lambda\lambda} < 0$$

for all $t < t'$ and all unit vector λ . In other words, t' is the first time (5.1) fails. We consider two cases, (\mathbf{x}', t') is an interior point or a boundary point. But first, note that we have the evolution equation for $V = v + f_{\lambda'\lambda'}$

$$(5.2) \quad V_t = a^{ij} V_{ij} + (f_{ij} D a^{ij}) \cdot DV + (2f_{\lambda'ij} D a^{ij} + f_{ij} D(a^{ij})_{\lambda'}) \cdot Df_{\lambda'} - Cv.$$

If (\mathbf{x}', t') is an interior point, then because it is a maximum point of V in $\Omega_{t'}$, we have

$$\begin{aligned} a^{ij} V_{ij} &\leq 0 \\ DV &= 0. \end{aligned}$$

Substitute into (5.2) we have

$$V_t \leq (2f_{\lambda'ij} D a^{ij} + f_{ij} D(a^{ij})'_{\lambda'}) \cdot Df'_{\lambda'} - Cv.$$

Because at the point (\mathbf{x}', t')

$$v + f_{\lambda'\lambda'} \geq v + f_{\lambda\lambda}$$

or

$$f_{\lambda'\lambda'} \geq f_{\lambda\lambda}$$

for any other unit vector λ , we have

$$f_{\lambda\lambda'} = 0$$

for any $\lambda \perp \lambda'$. Hence,

$$\begin{aligned} V_t &\leq ((2f_{\lambda'ij} D a^{ij} + f_{ij} D(a^{ij})_{\lambda'}) \cdot \lambda') f_{\lambda'\lambda'} - Cv \\ &< -C f_{\lambda'\lambda'} - Cv \quad (\text{remember } f_{\lambda'\lambda'} = -v < 0) \\ &= 0 \end{aligned}$$

which contradicts the assumption that (\mathbf{x}', t') is the first time $V = 0$. So (\mathbf{x}', t') cannot be an interior point.

If \mathbf{x}' is on $\partial\Omega_{t'}$. Again, denote by ν the inward normal unit vector to $\partial\Omega_{t'}$ at x' . Then at this point we have from definition of (\mathbf{x}', t') and λ' ,

$$\begin{aligned} (v + f_{\lambda'\lambda'})_{\nu} &\leq 0 \\ f_{\nu\lambda'\lambda'} &\leq -v_{\nu} < 0. \end{aligned}$$

We will show that on the other hand

$$(5.3) \quad f_{\nu\lambda'\lambda'} = 0.$$

We have from the Lemma 4.2 that

$$f_{\nu\nu} < 0.$$

We also have as a consequence of the fact that $|Df| = 1$ on the free-boundary and $|Df| < 1$ in the interior that $f_{\nu\lambda} = 0$ for any tangential unit vector λ . Hence as a consequence of the fact $f_{\lambda'\lambda'} = 0$, λ' must be a tangential vector of $\partial\Omega_{t'}$. Otherwise, there would be a tangential vector λ that lies on the same plane with ν and λ' such that

$$f_{\lambda\lambda} > 0$$

which contradicts our assumption on (\mathbf{x}', t') and λ' .

Without loss of generality, we can assume that $\nu = e_1$ and $\lambda' = e_2$. Because e_1 is the unit normal vector of $\partial\Omega_{t'}$ at x' , in a small neighborhood of \mathbf{x}' , we can write $\partial\Omega_{t'}$ as the graph of a smooth function

$$x_1 = \gamma(x_2, x')$$

where $x' = (x_3, \dots, x_n)$. From here to the end of the proof, we will use γ' and γ'' to denote the first and second derivatives of γ with regards to x_2 . Differentiate $f = 0$ with regards to e_2 we have

$$f_1\gamma' + f_2 = 0$$

or $\gamma' = 0$ since $f_1 = 1$, $f_2 = 0$. Differentiate one more time and disregard all terms containing γ' we have

$$f_1\gamma'' + f_{22} = 0$$

and so $\gamma'' = 0$ since $f_{22} = 0$ due to our assumption. Differentiate $Df \cdot Df = 1$ twice with regards to e_2 and disregard all terms containing γ' or γ'' we obtain

$$Df \cdot Df_{22} + Df_2 \cdot Df_2 = 0$$

or

$$f_{122} + |Df_2|^2 = 0.$$

As above, because $f_{ii} \leq 0 = f_{22} \forall i$, we have

$$f_{2i} = 0$$

for all $i \neq 2$. But $f_{22} = 0$ as well, so $Df_2 = 0$. Hence

$$f_{122} = 0$$

which is exactly what we want to show in (5.3). We then have a contradiction. In other words

$$f_{\lambda\lambda} < 0$$

for all $(x, t) \in \overline{\Omega} \cap \{0 < t < T\}$ and all unit vector λ or equivalently, Ω_t is strictly convex and $f(\cdot, t)$ is strictly concave in Ω_t for all $t \in [0, T)$. \square

6. REGULARITY NEAR THE FREE-BOUNDARY

In this section, we show that the degeneracy $|Df| = 0$ is kept away from the free-boundary. Consequently, the free-boundary is smooth, uniformly in ϵ . It enables us to show that the limiting function obtained by letting ϵ go to 0 satisfies the boundary condition of the original problem. The proof depends crucially on the concavity of f .

We introduce some notations. We denote by $B_r(\mathbf{x})$ the disk of radius r around x

$$B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n \mid |\mathbf{y} - \mathbf{x}| < r\}$$

when $\mathbf{x} \in \mathbb{R}^n$ and $r \in \mathbb{R}$. We write B_r for $B_r(0)$. We also define

$$A_r = \{(\mathbf{x}, t) \mid \text{dist}(\mathbf{x}, \partial\Omega_t) < r\} \cap \Omega.$$

For any point $\mathbf{x} = (x_1, x_2, \dots, x_n)$, we define

$$\begin{aligned}\psi_1(\mathbf{x}) &= x_1 \\ \psi'(\mathbf{x}) &= (x_2, \dots, x_n).\end{aligned}$$

Lemma 6.1. *Assume all hypotheses as in the Lemma 5.1. Assume also that there exist positive numbers r, R and m and a point \mathbf{x}_0 such that*

$$(6.1) \quad B_r(\mathbf{x}_0) \subset \Omega_t \subset B_R(\mathbf{x}_0) \quad \text{for all } t \in [0, T)$$

$$(6.2) \quad f(\mathbf{x}, t) > m \quad \text{for all } (\mathbf{x}, t) \in B_r(\mathbf{x}_0) \times [0, T).$$

Then for any $0 < T_1 < T$ and $k \in \mathbb{Z}^+$, there exist positive numbers $d(r, R, m, T_1)$ and $C(d, k)$ such that

$$|f(\cdot, t)|_{C^k(A_d \cap \{T_1 \leq t < T\})} < C.$$

Proof. To simplify the notation, we assume that the conditions (6.1) holds for $\mathbf{x}_0 = 0$. In other words

$$\begin{aligned}B_r &\subset \Omega_t \subset B_R \quad \text{for all } t \in [0, T) \\ f(\mathbf{x}, t) &> m \quad \text{for all } (\mathbf{x}, t) \in B_r \times [0, T).\end{aligned}$$

Let (P, t) be a point on $\partial\Omega$ for some $t \in [T_1, T)$. Fix this value of t from here until the end of this proof. Without loss of generality, we can assume that

$$\begin{aligned}\psi_1(P) &< 0 \\ \psi'(P) &= 0\end{aligned}$$

First, we will show that $f_1(\mathbf{x}, t)$ is bounded away from 0 in a neighborhood of P in Ω_t . Consider any point Q in Ω_t that satisfies the following conditions

$$\begin{aligned}\psi_1(Q) &< 0 \\ |\psi'(Q)| &< r \\ f(Q, t) &< m/2.\end{aligned}$$

Let $R = (0, \psi'(Q))$. Because $R \in B_r$, we have $f(R, t) > m$. We also have $f(Q, t) < m/2$ and $f_1(\mathbf{x}, t)$ decreases in x_1 as a consequence of concavity (here f_1 denotes the first derivative of f with regards to x_1). Thus,

$$\begin{aligned}
m/2 &< f(R, t) - f(Q, t) \\
&= \int_{\psi_1(Q)}^0 f_1((x_1, \psi'(Q)), t) dx_1 \\
&\leq |\psi_1(Q)| f_1(Q, t) \\
&\leq R f_1(Q, t) \\
f_1(Q, t) &\geq m/2R
\end{aligned}$$

We just showed that if \mathbf{x} satisfies

$$\begin{aligned}
\psi_1(\mathbf{x}) &< 0 \\
|\psi'(\mathbf{x})| &< r \\
f(\mathbf{x}, t) &< m/2
\end{aligned}$$

then

$$f_1(\mathbf{x}, t) \geq m/2R.$$

On the set containing all such \mathbf{x} , the Implicit Function Theorem says that there exists a function g defined on the set

$$\mathcal{B} = \{(y, x') \in \mathbb{R} \times \mathbb{R}^{n-1} \mid 0 \leq y < m/2, |x'| < r\} \times [0, T)$$

such that

$$\psi_1(\mathbf{x}) = g(f(\mathbf{x}, t), \psi'(\mathbf{x})).$$

We will compute explicitly the evolution equation and boundary condition of g

$$\begin{aligned}
f_1 &= \frac{1}{g_1} \\
f_i &= -\frac{g_i}{g_1} \\
f_t &= -\frac{g_t}{g_1} \\
f_{11} &= -\frac{g_{11}}{g_1^3} \\
f_{1i} &= -\frac{g_1 g_{1i} - g_i g_{11}}{g_1^3} \\
f_{ij} &= -\frac{g_1^2 g_{ij} - g_1 g_j g_{1i} - g_1 g_i g_{1j} + g_i g_j g_{11}}{g_1^3}.
\end{aligned}$$

The boundary condition $|Df| = 1$ on $\partial\Omega_t$ is equivalent to

$$g_1 = \left(1 + \sum_{i=2}^n g_i^2\right)^{1/2}$$

on

$$\{(y, x') \in \mathbb{R} \times \mathbb{R}^{n-1} \mid y = 0, |x'| < r\} \times [0, T).$$

Next we compute the evolution for g on \mathcal{B} . In all \sum appearing in the following computations, unless explicitly marked otherwise, indices i and j run from 2 to n . Let

$$M = 1 + \sum g_i^2.$$

Then

$$\begin{aligned} |Df|^2 &= \frac{M}{g_1^2} \\ \Delta f &= -\frac{1}{g_1^3} \left(g_{11} + g_1^2 \sum g_{ii} - 2g_1 \sum g_i g_{1i} + g_{11} \sum g_i^2 \right) \\ &= -\frac{1}{g_1^3} \left(M g_{11} + g_1^2 \sum g_{ii} - 2g_1 \sum g_i g_{1i} \right) \\ f_i f_j f_{ij} &= -\frac{1}{g_1^5} \left(g_{11} - 2 \sum g_i (g_1 g_{1i} - g_i g_{11}) \right. \\ &\quad \left. + \sum g_i g_j (g_1^2 g_{ij} - g_1 g_i g_{1j} - g_1 g_j g_{1i} + g_i g_j g_{11}) \right) \\ &= -\frac{1}{g_1^5} \left(g_{11} M^2 - 2g_1 M \sum g_i g_{1i} + g_1^2 \sum g_i g_j g_{ij} \right). \end{aligned}$$

Substitute into the equation for f_t ,

$$\begin{aligned} g_t &= \frac{(M + \epsilon)^{q-1}}{g_1^{2q}} \left(M g_{11} + g_1^2 \sum g_{ii} - 2g_1 \sum g_i g_{1i} \right) \\ &\quad + 2(q-1) \frac{(M + \epsilon)^{q-2}}{g_1^{2q}} \left(g_{11} M^2 - 2g_1 M \sum g_i g_{1i} + g_1^2 \sum g_i g_j g_{ij} \right) \\ &= b^{ij} g_{ij}. \end{aligned}$$

We want to show that there exist positive numbers λ , Λ , independent of ϵ and t such that

$$\lambda \leq b^{ij} \xi_i \xi_j \leq \Lambda$$

for all unit vector $\xi \in \mathbb{R}^n$. First, because

$$1 \geq |Df| \geq f_1 \geq m/2R,$$

we have

$$1 \leq M^{1/2} \leq g_1 \leq 2R/m.$$

The upper bound Λ then is obvious. For the lower bound, because

$$M^2 \xi_1^2 - 2g_1 M \sum g_i \xi_1 \xi_i + g_1^2 \sum g_i g_j \xi_i \xi_j = (M \xi_1 - \sum g_i \xi_i)^2,$$

it is enough to show that

$$M\xi_1^2 - 2g_1 \sum g_i \xi_1 \xi_i + g_1^2 \sum \xi_i^2 \geq \lambda$$

for some positive λ . We have

$$\left(g_1^2 - \frac{1}{2n}\right) \left(g_i^2 + \frac{1}{2n}\right) - (g_1 g_i)^2 = \frac{1}{2n} \left(g_1^2 - g_i^2 - \frac{1}{2n}\right) \geq 0$$

since $g_1^2 \geq M = 1 + \sum g_i^2$ and so

$$\left(\frac{1}{2n} + g_i^2\right) \xi_1^2 - 2g_1 g_i \xi_1 \xi_i + \left(g_1^2 - \frac{1}{2n}\right) \xi_i^2 \geq 0 \quad \text{for all } 2 \leq i \leq n.$$

Summing up we obtain

$$\begin{aligned} \left(\frac{n-1}{2n} + \sum g_i^2\right) \xi_1^2 - 2g_1 \sum g_i \xi_1 \xi_i + \left(g_1^2 - \frac{1}{2n}\right) \sum \xi_i^2 &\geq 0 \\ M\xi_1^2 - 2g_1 \sum g_i \xi_1 \xi_i + g_1^2 \sum \xi_i^2 &\geq \frac{1}{2n}. \end{aligned}$$

From the theory of quasi-linear parabolic equation with oblique boundary condition we can choose $d < \min(r, m/2)$ such that $g(., t)$ is in C^∞ on the set

$$\{(y, x') \in \mathbb{R} \times \mathbb{R}^{n-1} \mid 0 \leq y \leq d, |x'| \leq d\} \times [T_1, T)$$

and for any k , the norm $|g(., t)|_{C^k}$ depends only on k, d, r, R, m and T_1 , not on ϵ, t or g_0 . Revert back to f , we conclude that f is smooth on the set

$$\{(\mathbf{x}, t) \in \Omega \mid |\psi'(\mathbf{x})| \leq d, \psi_1(\mathbf{x}) < 0, f(\mathbf{x}, t) \leq d\} \cap \{T_1 \leq t < T\}$$

and again, for any k , the norm $|f(., t)|_{C^k}$ on this set depends only on k, d, r, R, m and T_1 . Note that the above set includes the set

$$B(P, d) \cap \Omega_t.$$

The conclusion is of course true for any point on $\partial\Omega_t$ in place of P where $t \in [T_1, T)$. The lemma then follows. \square

7. COMPARISON PRINCIPLE

In the first lemma here, we show that if f'_0 is strictly greater than f_0 , then a solution to the problem $(P(\epsilon))$ with initial value f'_0 remains strictly greater than a solution with initial value f_0 .

Lemma 7.1. *Suppose that f and f' are solutions up to some finite time T to the problem $P(\epsilon)$ and $P(\epsilon')$ respectively for some $\epsilon \geq \epsilon' > 0$. Suppose also that at the time $t = 0$,*

$$\begin{aligned} \overline{\Omega_0} &\subset \Omega'_0 \\ f_0(\mathbf{x}) &< f'_0(\mathbf{x}) \quad \text{in } \overline{\Omega_0}. \end{aligned}$$

Then for any $t \in (0, T)$,

$$\begin{aligned} \overline{\Omega}_t &\subset \Omega'_t \\ f(\mathbf{x}, t) &< f'(\mathbf{x}, t) \quad \text{in } \overline{\Omega}_t. \end{aligned}$$

Proof. Since

$$f_0(\mathbf{x}) < f'_0(\mathbf{x}) \quad \text{in } \overline{\Omega}_0$$

we can choose a positive number m such that

$$f_0(\mathbf{x}) + m < f'_0(\mathbf{x}) \quad \text{in } \overline{\Omega}_0.$$

Choose a positive number δ such that $\delta T < m$. We will show that

$$\begin{aligned} \overline{\Omega}_t &\subset \Omega'_t \\ f'(\mathbf{x}, t) - f(\mathbf{x}, t) - m + \delta t &> 0 \quad \text{in } \overline{\Omega}_t \end{aligned}$$

for all $t \in [0, T)$. Assuming it is not the case, there must be a first time t_0 such that at least one of the two above conditions is violated. Assume that the first condition is violated at t_0 . In other words, $\partial\Omega_t$ and $\partial\Omega'_t$ touches at some point \mathbf{x}_0 , then at that point

$$f'(\mathbf{x}_0, t_0) - f(\mathbf{x}_0, t_0) - m + \delta t_0 = -m + \delta t_0 < 0$$

which implies that the second condition must be violated before time t_0 , contradicting our choice of t_0 . Hence

$$\overline{\Omega}_t \subset \Omega'_t$$

for all $t \in [0, t_0]$. The second condition is violated implies that there is a point $\mathbf{x}_0 \in \overline{\Omega}_{t_0}$ such that

$$f'(\mathbf{x}_0, t_0) - f(\mathbf{x}_0, t_0) - m + \delta t_0 = 0.$$

We consider the case $\mathbf{x}_0 \in \partial\Omega_{t_0} \subset \Omega'_{t_0}$ first. Let ν be the inward unit normal to $\partial\Omega_{t_0}$ at \mathbf{x}_0 . From the definition of (\mathbf{x}_0, t_0) we must have

$$\begin{aligned} (f'(\mathbf{x}_0, t_0) - f(\mathbf{x}_0, t_0) - m + \delta t_0)_\nu &\geq 0 \\ f'_\nu(\mathbf{x}_0, t_0) - f_\nu(\mathbf{x}_0, t_0) &\geq 0 \\ f'_\nu(\mathbf{x}_0, t_0) &\geq 1 \end{aligned}$$

which contradicts the result of Lemma 4.1.

If $\mathbf{x}_0 \in \Omega_{t_0} \subset \Omega'_{t_0}$, then because it is an minimum point for $f' - f$ on Ω_{t_0} , we have

$$\begin{aligned} Df'(\mathbf{x}_0, t_0) &= Df(\mathbf{x}_0, t_0) \\ 0 &\geq \Delta f'(\mathbf{x}_0, t_0) \geq \Delta f(\mathbf{x}_0, t_0) \\ 0 &\geq f'_{\nu\nu}(\mathbf{x}_0, t_0) \geq f_{\nu\nu}(\mathbf{x}_0, t_0). \end{aligned}$$

Plug into the equation for f'_t and f_t , recalling that $\epsilon \geq \epsilon'$ we obtain

$$\begin{aligned} f'_t &= (|Df'|^2 + \epsilon')^{q-1} \Delta f' + 2(q-1)(|Df'|^2 + \epsilon')^{q-2} |Df'|^2 f'_{\nu\nu} \\ &\geq (|Df|^2 + \epsilon)^{q-1} \Delta f + 2(q-1)(|Df|^2 + \epsilon)^{q-2} |Df|^2 f_{\nu\nu} \\ &= f_t. \end{aligned}$$

On the other hand, because t_0 is the first time $f' - f - m + \delta t = 0$,

$$f'_t - f_t + \delta \leq 0.$$

Again, we arrive a contradiction. In other words,

$$\begin{aligned} \overline{\Omega}_t &\subset \Omega'_t \\ f' - f - m + \delta t &> 0 \end{aligned}$$

for all $t \in [0, T)$. The Lemma then follows readily. \square

Remark 7.1. If we let

$$\begin{aligned} m &\rightarrow \min \{f'_0(\mathbf{x}) - f_0(\mathbf{x}) \mid \mathbf{x} \in \overline{\Omega}_0\} \\ \delta &\rightarrow 0 \end{aligned}$$

we actually prove that

$$m(t) = \min \{f'(\mathbf{x}, t) - f(\mathbf{x}, t) \mid \mathbf{x} \in \overline{\Omega}_t\}$$

is a non-decreasing function.

We prove a slightly improved version of the last lemma.

Lemma 7.2. *Suppose f and f' are solutions up to time T to the problem $P(\epsilon)$ and $P(\epsilon')$ respectively for some $\epsilon \geq \epsilon' > 0$. Suppose also that at the time $t = 0$,*

$$\begin{aligned} \Omega_0 &\subset \Omega'_0 \\ f_0(\mathbf{x}) &\leq f'_0(\mathbf{x}) \quad \text{in } \Omega_0. \end{aligned}$$

Then for any $t \in [0, T)$,

$$\begin{aligned} \Omega_t &\subset \Omega'_t \\ f(\mathbf{x}, t) &\leq f'(\mathbf{x}, t) \quad \text{in } \Omega_t. \end{aligned}$$

Proof. Without loss of generality, we can assume that f_0 attains its maximum value at the origin. For each positive λ , define

$$\begin{aligned} \Omega^\lambda &= \{(\mathbf{x}, t) \mid (\lambda\mathbf{x}, \lambda^2 t) \in \Omega\} \\ f^\lambda &= \frac{1}{\lambda} f(\lambda\mathbf{x}, \lambda^2 t) \\ \Omega_0^\lambda &= \{\mathbf{x} \mid \lambda\mathbf{x} \in \Omega_0\} \\ f_0^\lambda &= \frac{1}{\lambda} f_0(\lambda\mathbf{x}, \lambda^2 t). \end{aligned}$$

It is clear that f^λ is a solution to the problem $(P(\epsilon))$ with respect to the initial data f_0^λ . Furthermore, since f_0 is concave, for each $\lambda > 1$ we have

$$\begin{aligned} \overline{\Omega_0^\lambda} &\subset \Omega_0 \subset \Omega'_0 \\ f_0^\lambda &< f_0 \leq f'_0 \quad \text{in } \Omega_0^\lambda. \end{aligned}$$

The last lemma says that for all $t \in [0, T/\lambda^2)$,

$$\overline{\Omega_t^\lambda} \subset \Omega'_t$$

$$\frac{1}{\lambda} f(\lambda \mathbf{x}, \lambda^2 t) = f^\lambda(\mathbf{x}, t) < f'(\mathbf{x}, t) \quad \text{in } \Omega_t^\lambda.$$

Let $\lambda \rightarrow 1$ we obtain

$$\Omega_t \subset \Omega'_t$$

$$f(\mathbf{x}, t) \leq f'(\mathbf{x}, t) \quad \text{in } \Omega_t$$

for all $t \in [0, T)$. □

Corollary 7.3. *Suppose (f, Ω) and (f', Ω') are two solutions to the problem $(P(\epsilon))$ with respect to the same initial value f_0 up to time T . Then $f = f'$ on $\mathbb{R}^n \times [0, T)$.*

8. LONG-TIME EXISTENCE FOR THE REGULARIZED PROBLEM

Lemma 8.1. *Assume that f_0 satisfies all hypotheses of the Lemma 3.1. Then there exists a unique solution to the problem $P(\epsilon)$ up to some positive time $T > 0$ where*

$$\lim_{t \rightarrow T} f(\mathbf{x}, t) = 0$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Proof. Let T be the maximal existence time for solutions to the problem $P(\epsilon)$ with the initial data f_0 . Due to the uniqueness result in section 10, there must be a solution f that exists up to time T . From the short-time existence result, T must be positive. We will show that

$$\lim_{t \rightarrow T} f(\mathbf{x}, t) = 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Assuming otherwise, then the same argument in the proof for the Lemma 11.1 can be used to show that T must be finite. In other words, due to the concavity of f_0 , there exists a number $c < 0$ such that

$$\operatorname{div} ((|Df_0|^2 + \epsilon)^{q-1} Df_0) < c \quad \text{in } \Omega_0$$

and consequently,

$$T \leq \frac{\max f_0}{|c|}.$$

We will prove that we can then extend this solution to a time $T' > T$. From the concavity of $f(\cdot, t)$, f is a decreasing function in t . Define

$$f_T(\mathbf{x}) = \lim_{t \rightarrow T} f(\mathbf{x}, t).$$

Since $|Df| \leq 1$ in Ω , f_T is continuous. Because f_T is not identically 0, there exist a ball $B_r(\mathbf{x}')$ and a positive number m such that

$$f_T > m \quad \text{in } B_r(\mathbf{x}').$$

From the Lemma 6.1, for all $t \in [T/2, T)$, there exists a positive number d such that f is smooth up to the boundary and time T in the set

$$\{(\mathbf{x}, t) \mid \text{dist}(\mathbf{x}, \partial\Omega_t) < d\} \cap \Omega_{[T/2, T)}.$$

Combine with the smoothness (depending on ϵ) of f up to time T in the interior of $\Omega_{[T/2, T)}$ from the standard theory of parabolic equation, we obtain the smoothness up to the boundary and time T of f in $\Omega_{[T/2, T)}$. Consequently, f_T is smooth up to the boundary. From the Lemma 5.1, we know that Ω_T is convex and f_T is concave in Ω_T . However, we need a stronger result that Ω_T is strictly convex and f_T is strictly concave in Ω_T in order to apply the Lemma 3.1. In deed, we can improve the result in the lemma 5.1 by duplicating the proof and substituting T' by T directly. In that proof, because we did not have the smoothness of f up to time T , we need to introduce $T' < T$ to guarantee the existence of a finite number $C(T')$ such that

$$|2f_{\lambda ij}Da^{ij} + f_{ij}D(a^{ij})_\lambda| < C$$

for all $t \in [t, T']$. But now we have the smoothness of f up to time T , we can derive the fact that there exists a number $C(T)$ such that the above inequality holds for all $t \in [0, T)$. The proof then guarantees that f is strictly concave at the time T .

The function f_T now satisfies all hypotheses of the Lemma 3.1. By that Lemma, we can then extend the solution f to some time $T' > T$. It contradicts the maximality of T . So we must have

$$\lim_{t \rightarrow T} f(\mathbf{x}, t) = 0$$

for all $\mathbf{x} \in \mathbb{R}^n$. □

9. EXISTENCE OF SOLUTION TO THE p -LAPLACIAN PROBLEM

In this section, we will pass ϵ to 0 and obtain a solution to our degenerate problem.

Lemma 9.1. *Assume that Ω_0 is a bounded and convex domain. The function f_0 is positive and concave in Ω_0 . Furthermore, on the boundary $\partial\Omega_0$, f_0 satisfies*

$$\begin{aligned} f_0(\mathbf{x}) &= 0 \quad \text{for all } \mathbf{x} \\ |Df_0(\mathbf{x})| &= 1 \quad \text{for a.e. } \mathbf{x}. \end{aligned}$$

Then there exists a solution to the problem (P) up to some time T where

$$\lim_{t \rightarrow T} f(\mathbf{x}, t) = 0 \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

The free-boundary $\partial\Omega_t$ is smooth for all $t \in (0, T)$.

Proof. Choose a sequence of functions f_0^ϵ with positive sets Ω_0^ϵ for all $\epsilon \in (0, 1)$ such that

$$\begin{aligned} \Omega_0^\epsilon &\in C^\infty \text{ and } f_0^\epsilon \in C^\infty(\overline{\Omega_0^\epsilon}), \\ \Omega_0^\epsilon &\text{ is strictly convex,} \\ f_0^\epsilon &\text{ is strictly concave,} \\ \Omega_0^{\epsilon_1} &\subset \Omega_0^{\epsilon_2} \text{ and } f_0^{\epsilon_1} \leq f_0^{\epsilon_2} \text{ if } \epsilon_1 > \epsilon_2, \\ \Omega_0 &= \cup \Omega_0^\epsilon \text{ and } f_0(\mathbf{x}) = \lim_{\epsilon \rightarrow 0} f_0^\epsilon(\mathbf{x}) \text{ for all } x \in \mathbb{R}^n, \\ |Df^\epsilon| &= 1 \text{ on } \partial\Omega^\epsilon. \end{aligned}$$

In other words, f_0^ϵ is an increasing sequence of smooth and strictly concave function that converge to f_0 as $\epsilon \rightarrow 0$. From the Lemma 8.1, for each ϵ , there exists a unique solution f_ϵ to the problem (P(ϵ)) up to some time T^ϵ where it vanishes identically. We will prove that $\lim_{\epsilon \rightarrow 0} f^\epsilon$ is a solution to the original problem (P).

From the lemma 7.2 and our choice of f_0^ϵ , it is clear that if $\epsilon_1 > \epsilon_2$, then

$$\begin{aligned} T^{\epsilon_1} &\leq T^{\epsilon_2} \\ \Omega^{\epsilon_1} &\subset \Omega^{\epsilon_2} \\ f^{\epsilon_1} &\leq f^{\epsilon_2} \text{ in } \Omega^{\epsilon_1}. \end{aligned}$$

Define

$$\begin{aligned} T &= \lim_{\epsilon \rightarrow 0} T^\epsilon \\ \Omega &= \cup \Omega^\epsilon \\ f(\mathbf{x}, t) &= \lim_{\epsilon \rightarrow 0} f^\epsilon(\mathbf{x}, t) \text{ for } (\mathbf{x}, t) \in \mathbb{R}^n \times [0, T). \end{aligned}$$

Due to the uniform smoothness of f^ϵ in a neighborhood of $\partial\Omega^\epsilon$, we have

$$\begin{aligned} \Omega_{(0, T)} &\in C^\infty \\ f &= 0 \text{ and } |Df| = 1 \text{ on } \partial\Omega \times \{0 < t < T\}. \end{aligned}$$

If $(\mathbf{x}, t) \in \Omega$, there exists an ϵ_0 such that $(\mathbf{x}, t) \in \Omega^\epsilon$ for all $\epsilon < \epsilon_0$. Since $f^\epsilon(\mathbf{x}, t)$ increases as ϵ decreases to 0,

$$f(\mathbf{x}, t) = \lim_{\epsilon \rightarrow 0} f^\epsilon(\mathbf{x}, t) > 0.$$

From the bound $|Df| \leq 1$ and the Corollary 2.15 in Chapter II of [10], we have interior $C_t^{0, 1/2}$ estimate for f^ϵ as functions of t , uniformly in t and ϵ . Together with the fact that for every \mathbf{x} ,

$$\lim_{t \rightarrow T^\epsilon} f^\epsilon(\mathbf{x}, t) = 0$$

we have

$$\lim_{t \rightarrow T} f(\mathbf{x}, t) = 0.$$

Because for every ϵ and t

$$f^\epsilon(\mathbf{x}, t) \leq f^\epsilon(\mathbf{x}, 0) < f_0(\mathbf{x}, 0),$$

we have

$$f(\mathbf{x}, t) = \lim_{\epsilon \rightarrow 0} f^\epsilon(\mathbf{x}, t) \leq f_0(\mathbf{x}).$$

On the other hand, since $f^\epsilon(\mathbf{x}, t)$ increases as ϵ decreases to 0,

$$\lim_{t \rightarrow 0} f(\mathbf{x}, t) \geq \lim_{t \rightarrow 0} f^\epsilon(\mathbf{x}, t) = f_0^\epsilon(\mathbf{x})$$

for any ϵ . Consequently,

$$\lim_{t \rightarrow 0} f(\mathbf{x}, t) \geq \lim_{\epsilon \rightarrow 0} f_0^\epsilon(\mathbf{x}) = f_0(\mathbf{x}).$$

Hence

$$\lim_{t \rightarrow 0} f(\mathbf{x}, t) = f_0(\mathbf{x}).$$

From $|Df^\epsilon| \leq 1$, we can choose a sequence of ϵ converging to 0 such that

$$Df^\epsilon \rightharpoonup Df$$

in all compact subsets of Ω . Given any function $\theta \in C_0^\infty(\Omega)$ and $0 < t_1 < t_2 < T$ we have from the equation

$$f_t^\epsilon = \operatorname{div}((|Df^\epsilon|^2 + \epsilon)^{q-1} Df^\epsilon)$$

that

$$\int_{\Omega(t_1, t_2)} f^\epsilon \theta_t d\mathbf{x} dt - \int_{\Omega(t_1)} f^\epsilon \theta d\mathbf{x} \Big|_{\Omega_{t_1}}^{\Omega_{t_2}} = \int_{\Omega(t_1, t_2)} (|Df^\epsilon|^2 + \epsilon)^{q-1} Df^\epsilon \cdot D\theta d\mathbf{x} dt.$$

Passing to the limit we then obtain

$$\int_{\Omega(t_1, t_2)} f \theta_t d\mathbf{x} dt - \int_{\Omega(t_1)} f \theta d\mathbf{x} \Big|_{\Omega_{t_1}}^{\Omega_{t_2}} = \int_{\Omega(t_1, t_2)} |Df|^{2(q-1)} Df \cdot D\theta d\mathbf{x} dt.$$

□

10. UNIQUENESS

Lemma 10.1. *Solution obtained in the Lemma 9.1 is the unique solution to the problem (P).*

Proof. Assume that there exists another solution g to the problem (P). Let Ω^* be the positive set of g and T^* its existence time. Also without loss of generality, assuming that f_0 attains its maximum value at 0. For each positive λ , it is clear that

$$g^\lambda(\mathbf{x}, t) = \lambda^{-1} g(\lambda^2 \mathbf{x}, \lambda^{p+2} t)$$

is a solution to the problem $P(\epsilon)$ with positive set

$$\Omega^\lambda = \{(\mathbf{x}, t) \mid (\lambda^2 \mathbf{x}, \lambda^{p+2} t) \in \Omega^*\}$$

and initial data

$$g_0^\lambda(\mathbf{x}, t) = \lambda^{-1} f_0(\lambda^2 \mathbf{x}, \lambda^{p+2} t).$$

Clearly, for $\lambda < 1$,

$$\begin{aligned}\overline{\Omega_0} &\subset \Omega_0^\lambda \\ f_0 &< g_0^\lambda \quad \text{in } \Omega_0.\end{aligned}$$

We will show that for all $t < \min(T, \lambda^{-(p+2)}T^*)$,

$$\begin{aligned}\overline{\Omega_t} &\subset \Omega_t^\lambda \\ f(\mathbf{x}, t) &< g^\lambda(\mathbf{x}, t) \quad \text{in } \Omega_t.\end{aligned}$$

Assuming it is not the case, then there must be a first time t_0 where at least one of those two inequalities is violated. If the first one is violated at the time t_0 , it means $\partial\Omega_t$ and $\partial\Omega_t^\lambda$ touch at some point \mathbf{x}_0 . At that point (\mathbf{x}_0, t_0) ,

$$|Df| = 1 > \lambda = |Dg^\lambda|.$$

There must be then a point $\mathbf{x} \in \Omega_t$ such that

$$f(\mathbf{x}, t_0) > g^\epsilon(\mathbf{x}, t_0).$$

which implies that the second inequality must be violated at some time before t_0 . So, up to time t_0 ,

$$\overline{\Omega_t} \subset \Omega_t^\lambda.$$

Consequently, on the parabolic boundary of $\Omega_{[0, t_0]}$, $f < g^\lambda$. Thus, from the lemma 3.1 in Chapter VI of [6], we have $f < g$ in $\Omega_{[0, t_0]}$ which contradicts our choice of t_0 . Hence for all $t < \min(T, \lambda^{-(p+2)}T^*)$,

$$\begin{aligned}\overline{\Omega_t} &\subset \Omega_t^\lambda \\ f(\mathbf{x}, t) &< g^\lambda(\mathbf{x}, t) = \lambda^{-1}g(\lambda^2\mathbf{x}, \lambda^{p+2}t) \quad \text{in } \Omega_t.\end{aligned}$$

Let $\lambda \rightarrow 1$ we obtain for all $t < \min(T, T^*)$,

$$\begin{aligned}\Omega_t &\subset \Omega_t^* \\ f(\mathbf{x}, t) &\leq g(\mathbf{x}, t) \quad \text{in } \Omega_t.\end{aligned}$$

Arguing similarly for $\lambda > 1$ we on the other hand obtain

$$\begin{aligned}\Omega_t^* &\subset \Omega_t \\ g(\mathbf{x}, t) &\leq f(\mathbf{x}, t) \quad \text{in } \Omega_t^*.\end{aligned}$$

Thus for all $t < \min(T, T^*)$

$$\begin{aligned}\Omega_t &= \Omega_t^* \\ f(\mathbf{x}, t) &= g(\mathbf{x}, t).\end{aligned}$$

□

11. VANISHING IN FINITE TIME

Lemma 11.1. *The existence time T of the solution obtained in the Lemma 9.1 is finite.*

Proof. Clearly from the Comparison Principle in section 7 and scaling that it is enough to prove this lemma for one particular initial function f_0 . We show that if a smooth function f_0 satisfies all hypotheses of the Lemma 9.1 and

$$\Delta_p f_0 < c$$

for some $c < 0$, then for any $0 < t_1 < t_2 < T$ and any $\mathbf{x} \in \Omega_{t_1}$, the corresponding solution f satisfies the inequality

$$(11.1) \quad f(\mathbf{x}, t_2) - f(\mathbf{x}, t_1) \leq c(t_2 - t_1).$$

It then readily follows that

$$T \leq \frac{\max f_0}{|c|}.$$

Choose the sequence $\{f_0^\epsilon\}$ so that

$$\operatorname{div}((|Df_0^\epsilon|^2 + \epsilon)^{q-1} Df_0^\epsilon) < c.$$

We will show that f^ϵ satisfies

$$f_t^\epsilon \leq c$$

for all ϵ and (11.1) then follows immediately.

Differentiating the equation satisfied by f^ϵ with respect to t , it is easy to see that f_t^ϵ satisfies the Maximum Principle. Since $f_t^\epsilon \leq c$ at the time $t = 0$ from our choice of f_0^ϵ , all we need to show is that f_t^ϵ cannot attain its maximum value on the free-boundary. From the Hopf's Lemma, if f_t^ϵ attains its maximum value at a point \mathbf{x}_0 on the free-boundary $\partial\Omega_{t_0}$, then we must have

$$(f_\nu^\epsilon)_t(\mathbf{x}_0, t_0) = (f_t^\epsilon)_\nu(\mathbf{x}_0, t_0) < 0$$

where ν is the inward unit normal at \mathbf{x}_0 with respect to $\partial\Omega_{t_0}$. On the other hand, since Ω_t shrinks in time, for any $t < t_0$ $\mathbf{x}_0 \in \Omega_t$ and so,

$$f_\nu^\epsilon(\mathbf{x}_0, t) < 1$$

while

$$f_\nu^\epsilon(\mathbf{x}_0, t_0) = 1$$

which lead to

$$(f_\nu^\epsilon)_t(\mathbf{x}_0, t_0) \geq 0.$$

□

Remark 11.1. The finiteness for the existence time holds for any initial data with bounded support, not just concave ones.

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